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THE GAMMA FUNCTION WITH VARYING DIFFERENCE INTERVAL *

by

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Before actually discussing the gamma function it is necessary to introduce a section on general periodic functions and one on a generalization of the Euler-Maclaurin sum formula.

1. General Periodic Functions

Let $z = x + yi$ and $h(z) = p(z) + iq(z)$, where x, y, $p(z)$ and $q(z)$ are real. Then $f(z)$ is called general periodic with period $h(z)$, over a domain R, if

(1) $$f(z + h(z)) = f(z)$$

whenever z and $z + h(z)$ belong to R.

An example of a general periodic function is $\sin z^2$ where $h(z) = -z + \sqrt{z^2 + 2\pi}$. Any determination may be given to $\sqrt{z^2 + 2\pi}$. Other examples are easy to construct.

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We now place certain restrictions on $h(z)$ and under these restrictions show that general periodic functions with period $h(z)$, other than constants, exist. These restrictions are not necessary. The reader can formulate other sufficient conditions.

Assume $0 < \varepsilon < p(z)$ and that if z and $\bar{z}$ are different numbers in R then $z + h(z) \neq \bar{z} + h(\bar{z})$. We now draw the lines $x = \varepsilon$, $x = 2\varepsilon, \ldots, x = n\varepsilon \ldots$ in the complex plane. These lines divide the right-hand half of the complex plane into strips. We now assign values to $f(z)$ at all points of the strip $0 \leq x \leq \varepsilon$. From these values determine $f(z) + h(z)$ by equation (1). If $f(z)$ is not determined at all points of the strip $\varepsilon < x \leq 2\varepsilon$ assign it a value at each of these points as desired. Now determine $f(z)$ for the points of the strip $2\varepsilon < x \leq 3\varepsilon$ from its values on the strip $\varepsilon < x \leq 2\varepsilon$ by (1) so far as is possible and so far as $f(z)$ has not already been determined by the points of the strip $0 \leq x \leq \varepsilon$. If there exist now points of the strip $2\varepsilon \leq x < 3\varepsilon$ where $f(z)$ has not been defined we assign values to $f(z)$ at these points as desired. We now consider the strip $3\varepsilon \leq x < 4\varepsilon$. So far as $f(z)$ can not be defined at points of this strip from its values on the strip $0 \leq x < 3\varepsilon$ we assign $f(z)$ values at these points as desired. We continue to proceed in this manner until $f(z)$ is defined at all points of the half plane $x \geq 0$. It is immediate that $f(z)$ is general periodic with period $h(z)$ over this half plane. It is in general discontinuous.

2. The Euler-Maclaurin Formula

Let $h(x)$ be positive real and $x + h(x)$ monotonic increasing with x. Now let $x_1 = x$, $x_2 = x_1 + h(x_1)$, ..., $x_n = x_{n-1} + h(x_{n-1})$. Then let $x_n - x_1 = h_n(x)$. Then $h_n(x) = h_{n-1}(x) + h(x_{n-1})$ and $h_n(x) = \sum_{i=1}^{n-1} h(x_i)$. In the work which follows $B_n(x)$ is the Bernoulli polynomial of order n, also $\overline{B}_n(x)$ is the function with period 1 coinciding with $B_n(x)$ over the interval [0, 1]. We also let B_n be the Bernoulli ~~number~~ of order n and $Q_n(x) = B_n(x) - B_n$, $\overline{Q}_n(x) = \overline{B}_n(x) - B_n$. We remark that $\frac{d}{dx} B_n(x) = n\, B_{n-1}(x)$. For other properties of the Bernoulli polynomials and numbers see, for example, Fort, Chapter III. We now write down

$$(2) \qquad - R_m(a) = h^m(a) \int_0^1 \frac{\overline{B}_m(w - t)}{m!} F^{(m)}(a + h(a)t)dt.$$

Carrying out certain operations on this as explained in detail in Fort, page 51, we arrive at the formula,

$$(3) \qquad F(a + w\, h_1(a)) = \frac{1}{h(a)} \int_a^{a+h(a)} F(t)\, dt +$$

$$\sum_{v=1}^{m} h^{v-1}(a) \frac{B_v(w)}{v!} \Delta F^{(v-1)}(a)$$

$$- h^m(a) \int_0^1 \frac{\overline{B}_m(w-t)}{m!} F^{(m)}(a + h_1(a)t)\, dt.$$

Here $\Delta F^{(v-1)}(a) = F^{(v-1)}(a + h(a) - F^{(v-1)}(a)$

The parenthetical superscripts denote differentiation. Formula (3) is the basic Euler-Maclaurin formula.

Now let $w = 0$, $m = 2k$. Note that $Q_{2k}(1 - t) = Q_{2k}(t)$ and that $B_{2k-1} = 0$ when $k > 1$. We then let a equal successively $x_1, x_2, \ldots, x_n$ and sum. We get, letting $k > 1$

$$(4) \quad \sum_{i=1}^{n} F(x_i)h(x_i) = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} F(t)dt + \sum_{i=1}^{n} \sum_{v=1}^{2k-2} \frac{B_v}{v!} \Delta F^{(v-1)}(x_i)h^v(x_i)$$

$$- \frac{1}{(2k)!} \int_0^1 [Q_{2k}(t) \sum_{i=1}^{n} F^{(2k)}(x_i + h(x_i)t)h^{2k+1}(x_i)]dt.$$

Denote the last expression by $-R^1_{2k}$. We have

$$(5) \quad R^1_{2k} = - \frac{1}{(2k)!} \int_0^1 [Q_{2k}(t) \sum_{i=1}^{n} F^{(2k)}(x_i + h(x_i)t)h^{2k+1}(x_i)]dt$$

Since $Q_{2k}(t)$ retains the same sign over the interval (0, 1) we can apply the first law of the mean for integrals. We get

$$R^1_{2k} = [- \int_0^1 Q_{2k}(t)dt] \frac{1}{(2k)!} \sum_{i=1}^{n} F^{(2k)}(x_i + \theta(x,n)h(x_i))h^{2k+1}(x_i),$$

$0 < \theta < 1$. But $Q_{2k}(t) = \frac{1}{2k+1} Q'_{2k+1}(t) - B_{2k}$

Hence $\int_0^1 Q_{2k}(t)dt = - B_{2k}$. Hence

$$(6) \quad R^1_{2k} = \frac{B_{2k}}{(2k)!} \sum_{i=1}^{n} F^{(2k)}(x_i + \theta(x,n)h(x_i))h^{2k+1}(x_i)$$

Since $\frac{B_{2k}}{2k!}$ is bounded

$$\left| R^1_{2k} \right| < M \sum_{i=1}^{n} \left| F^{(2k)}(a + \theta(x,n)h(x_i))h^{2k+1}(x_i) \right.$$

Under certain conditions we can obtain other forms for R^1_{2k}. Let us assume that $F^{(j)}(t)$ retains the same sign when $t > 0$ and that $F^{2j}(t).F^{2j-2}(t) > 0$ when $t > 0$ and $j = 2, 3, \ldots$. We note that

$Q'_{2k}(t) = 2k\, Q_{2k-1}(t)$, $k > 1$ and that

$Q'_{2k-1}(t) = (2k - 1)\, Q_{2k-2} - B_{2k-2}$ and that

$Q_{2k}(t).Q_{2k-2}(t) < 0$, $0 < t < 1$. We now consider (5) and integrate by parts twice. We obtain

$$(7)\quad R^1_{2k} = -\frac{B_{2k-2}}{(2k-2)!} \sum_{i=1}^{n} \Delta F^{(2k-3)}(x_i)h^{2k-2}(x_i)$$

$$-\frac{1}{(2k-2)!} \int_0^1 \Big[Q_{2k-2}(t) \sum_{i=1}^{n} F^{(2k-2)}(x_i + h(x_i)t)h^{2k-1}(x_i) \Big] dt$$

Now if $A = B + C$ and $AC < 0$ then $A = \theta B$, $0 < \theta < 1$. Hence

$$(8)\quad R^1_{2k} = \frac{B_{2k-2}}{(2k-2)!}\, \theta \sum_{i=1}^{n} (\Delta F^{(2k-3)}(x_i))h^{2k-2}(x_i).$$

We, of course, assume the existence of all derivatives that enter any formula.

If we advance k by 1 we have

$$(9)\quad R^1_{2k+2} = \frac{B_{2k}}{2k!}\, \theta \sum_{i=1}^{n} (\Delta F^{(2k-1)}(x_i))h(x_i)^{2k}$$

3. The Gamma Function

We shall solve the difference equation

$$(10) \qquad \frac{\Delta u(x)}{h(x)} = \ln x, \qquad x \geqq \delta > 0$$

Here $\Delta u(x) = u(x + h(x)) - u(x)$

We shall require * that $0 < c < h(x) < E$ where c and E are

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* A function that satisfies these conditions is

$$h(x) = 1 + \frac{1}{x}, \quad 0 \leqq \varepsilon \leqq x$$

To show that $\sum_{i=1}^{\infty} (\ln x_i) \Delta h(x_i)$ converges note that

$x + i - 1 < x_i < x + i - 1 + \frac{i - 1}{x}$. Hence

$$\Delta h(x_i) = \frac{h(x_i)}{x_i(x_i + h(x_i))} < \frac{1 + \frac{1}{c}}{(x + i - 1)\left(x + i - 1 + \frac{1}{x+i-1 + \frac{i-1}{x}}\right)}$$

$$< \frac{1 + \frac{1}{c}}{(x + i - 1)^2} .$$

$$\ln x_i < \ln\left(x + i - 1 + \frac{i - 1}{x}\right) = \ln\left(x + (i-1)\left(1 + \frac{1}{x}\right)\right) \leqq \ln(x + 2(i - 1))$$

if $x \geqq 1$. Hence

$$(\ln x_i) \Delta h(x_i) < \left(1 + \frac{1}{c}\right) \frac{\ln(x + 2(i-1))}{(x + i - 1)^2} < \left(1 + \frac{1}{c}\right) \frac{1}{(x + i - 1)^{3/2}}$$

$< \left(1 + \frac{1}{c}\right) \frac{1}{(i - 1)^{3/2}}$. This is the general term of a convergent series of constants. Consequently $\sum_{i=1}^{\infty} (\ln x_i) \Delta h(x_i)$ converges uniformly, $x \geqq 1$.

constants. We shall also require that $\sum_{i=1}^{\infty} (\ell n\, x_i) \Delta h(x_i)$ converge uniformly $x \geq \varepsilon$. We note that

$$\Delta h(x_i) = h(x_i + h(x_i)) - h(x_i) = h(x_{i+1}) - h(x_i)$$

It will be noticed that, although the work is written for $\ell n\, x$, we can replace $\ell n\, x$ by $F(x)$ with only trivial modifications if we require that $F^{(2j)}(x)$ exist and retain the same sign when $x \geq 1$ and that $F^{(2j)}(x)\, F^{2j-2}(x) > 0$ when $1 \leq j \leq k$ and that $\sum_{i=1}^{\infty} F(x_i) \Delta h(x_i)$ converge uniformly when $x \geq \varepsilon$ and that $F^{(v)}(x)\, x^{(v-1)}$ approaches 0 when x becomes infinite, $v = 1, 2, \ldots$

<u>Theorem</u>: <u>If</u> $\sum_{i=1}^{\infty} (\ell n\, x_i) \Delta h(x_i)$ <u>converges and if</u> $0 < c < h'(x) < E$ <u>where</u> c <u>and</u> E <u>are constants, then</u>

$$(11) \quad \int_1^{x_{n+1}} \ell nt\, dt - \sum_{i=1}^{n} (\ell n\, x_i) h(x_i) - \tfrac{1}{2}(\ell n\, x_{n+1}) h(x_{n+1})$$

<u>has a limit as x becomes infinite and the first difference of this limit is</u> $(\ell n\, x) h(x)$.

We shall call this limit $\ell n\, G(x)$. We are to prove

$$\frac{\Delta \ell n\, G(x)}{h(x)} = \frac{\ell n\, G(x + h(x)) - \ell n\, G(x)}{h(x)} = \ell n\, x.$$

Proof: We rewrite the Euler formula, replacing $F(x)$ by $\ell n\, x$ and changing signs throughout the equation. We use formula (6) for the remainder.

$$(12) \quad \int_{x}^{x_{n+1}} \ln t \, dt - \sum_{i=1}^{n} (\ln x_i) h(x_i) = - \sum_{i=1}^{n} \sum_{v=1}^{2k-2} \frac{B_v}{v!} (\ln^{(v-1)} x_i) h^{v}(x_i)$$

$$- \frac{B_{2k}}{(2k)!} \sum_{i=1}^{n} \ln^{(2k)}(x_i + \theta h(x_i)) h^{2k+1}(x_i).$$

Consider the last sum which we call R'_{2k}. Now $\frac{B_{2k}}{(2k)!}$ is bounded,

$$\ln^{(2k)}(x_i + \theta h(x_i)) h^{2k+1}(x_i) = \frac{(2k-1)!\, h^{2k+1}(x_i)}{[x_i + \theta h(x_i)]^{2k}}$$

$$= (2k-1)! \left[\frac{h(x_i)}{x_i + \theta h(x_i)}\right]^{2k} h(x_i) < N \left[\frac{1}{x_i}\right]^{2k}$$

since $h(x)$ is bounded. Here N is a constant. Consequently

$$\frac{B_{2k}}{(2k)!} \sum_{i=1}^{\infty} (2k-1)! \left[\frac{h(x_i)}{x_i + \theta h(x_i)}\right]^{2k} h(x_i) < N \sum_{i=1}^{\infty} \frac{1}{x_i^{2k}}$$

We call attention to the inequality $0 < c < h(x) < E$. Whereupon

$$(13) \quad \sum_{i=1}^{\infty} \frac{1}{x_i^{2k}} < \sum_{i=1}^{\infty} \frac{1}{[x + (i-1)c]^{2k}}$$

This series converges when $2k > 1$, which we assume. It follows that when $n \to \infty$, R'_{2k} approaches a function of x defined by a uniformly convergent series in x when $x \geq \varepsilon > 0$. This function multiplied by x^{2k-1} approaches zero when x becomes infinite.

We now consider the sum

$$-S = S_1 + S_2 + S_3 + \dots + S_{2k-2}\ , \quad \text{where}$$

$$S_1 = \sum_{i=1}^{n} B_1 [\ln(x_i + h(x_i)) - \ln x_i] h'(x_i)$$

$$S_2 = \sum_{i=1}^{n} \frac{B_2}{2!} \left[\frac{1}{x_{i+1}} - \frac{1}{x_i}\right] h^2(x_i)$$

$$S_3 = \sum_{i=1}^{n} \frac{B_3}{3!} \left[\frac{-1}{x_{i+1}^2} + \frac{1}{x_i^2}\right] h^3(x_i)$$

$$S_4 = \sum_{i=1}^{n} \frac{B_4}{4!} \left[\frac{2!}{x_{i+1}^3} - \frac{2!}{x_i^3}\right] h^4(x_i)$$

. . . .

$$S_{2k-2} = \sum_{i=1}^{n} \frac{B_{2k-2}}{(2k-2)!} \left[\frac{(2k-4)!}{x_{i+1}^{2k-3}} - \frac{(2k-4)!}{x_\xi^{2k-3}}\right] h^{2k-2}(x_i)\ .$$

Now let n become infinite. Each of the rows above becomes an infinite series. Consider

$$S_2 = \sum_{i=1}^{\infty} \frac{B_2}{2!} \left[\frac{1}{\xi}\right]^2 h^3(x_i) \quad \text{where} \quad x_i < \xi < x_{i+1}$$

Since $0 < c < h'(x) < E$

$$\frac{h^3(x_i)}{\xi^2} < \frac{E^3}{[x + c(i-1)]^2} < \frac{E^3}{c^2} \cdot \frac{1}{(i-1)^2}$$

Consequently when n becomes infinite S_2 becomes a uniformly convergent series in x when $x \geq \varepsilon > 0$. Clearly $S_3 \ldots\ldots S_{2k}$ can be treated precisely as S_2 with the same result.

Now consider the first of the above sums, namely

$$S_1 = B_1 \sum_{i=1}^{n} [\ell n\, x_{i+1} - \ell n\, x_i] h(x_i)$$

$$= B_1 \sum_{i=1}^{n} (\Delta \ell n\, x_i) h(x_i).$$

Summation by parts yields.

$$S_1 = -\frac{1}{2} [(\ell n\, x_{n+1}) h(x_{n+1}) - (\ell n\, x) h(x) - \sum_{i=1}^{n} (\ell n\, x_{i+1}) \Delta h(x_i)$$

Let n become infinite. We transpose $-\frac{1}{2}(\ell n\, x_{n+1}) h(x_{n+1})$ in (12). The infinite series $\sum_{i=1}^{\infty} (\ell n\, x_{i+1}) \Delta h(x_i)$ converges since $\sum_{i=1}^{\infty} (\ell n\, x_i) \Delta h(x_i)$ converges by hypothesis and since on on account of the boundedness of h, $\frac{\ell n\, x_{i+1}}{\ell n\, x_i}$ approaches 1.

We now add $\int_1^x \ell nt\, dt$ to both sides of (12). All that remains in the right member of (12) approaches a limit as n becomes infinite. The left member is that given in the theorem. We have

$$\int_1^{x_{n+1}} \ell nt\, dt - \sum_{i=1}^{n} (\ell n\, x_i) h(x_i) - \tfrac{1}{2}(\ell n\, x_{n+1}) h(x_{n+1}) =$$

$$x \ell nx - x + 1 + \tfrac{1}{2}(\ell n\, x) h(x) + \frac{1}{2} \sum_{i=1}^{n} (\ell n\, x_{i+1}) \Delta h(x_{i+1})$$

$$- \sum_{i=1}^{n} \sum_{v=2}^{2k-2} \frac{B_v}{v!} [\Delta \ell n^{(v-1)}\, x_i] h^v(x_i) + R'_{2k}, \quad \text{where}$$

$$R_{2k} = \frac{B_{2k}}{2k!} \sum_{i=1}^{n} [\ell n^{(2k)}(x_i + \theta h(x_i))] h^{2k+1}(x_i).$$

We now shall show that $\ell n\, G(x)$ satisfies (10). Consider (11) which we treat in three parts.

(a) $\Delta \int_1^{x_{n+1}} \ell nt\, dt = \int_{x_{n+1}}^{x_{n+2}} \ell nt\, dt = (x_{n+2} - x_{n+1}) \ell n\, \xi$,

where $x_{n+1} < \xi < x_{n+2}$

(b) $\Delta\, (- \sum_{i=1}^{n} (\ell n\, x_i) h(x_i) = - (\ell n\, x_{n+1}) h(x_{n+1}) + (\ell n\, x_1) h(x_1)$

(c) $\Delta \left[- \frac{1}{2} (\ell n\, x_{n+1}) h(x_{n+1}) \right] = - \frac{1}{2} [(\ell n\, x_{n+1}) \Delta\, h(x_{n+1}) +$

$+ h(x_{n+2})\ \Delta \ell n\, x_{n+1}\ ]$

We now consider these three results. From (a) and (b) $(\ell n\, \xi - \ell n\, x_{n+1}) h(x_{n+1}) = \frac{1}{\eta} (\xi - x_{n+1}) h(x_{n+1})$, where $x_{n+1} < \eta < \xi$. Since $h'(x)$ is bounded and $\xi - x_{n+1} < h(x_{n+1})$ This approaches zero. Moreover from (c) $(\ell n\, x_{n+1}) \Delta\, h(x_{n+1})$ approaches zero since it is the general term of a convergent series. Also $h(x_{n+2}) \Delta \ell\, n(x_{n+1})$ approaches zero since $h'(x)$ is bounded and $\Delta\, (\ell\, n\, x_{n+1}) = (x_{n+2} - x_{n+1}) \frac{1}{\xi} = h(x_{n+1}) \cdot \frac{1}{\xi}$. We now total what we have obtained from (a), (b) and (c). All that we get is $\ell n\, x_1\, h(x_1)$. But $x_1 = x$. Hence the theorem is proved.

4. Asymptotic Form

If we refer to formula (12) with the addition of $\int_1^x \ln t\, dt$ to both sides and the transposition of $\frac{1}{2}(\ln x_{n+1})h(x_{n+1})$ to the left member and then let n become infinite we have

$$(14)\quad \ln G(x) = x\ln x - x + 1 + \frac{1}{2}(\ln x)h(x) +$$

$$\frac{1}{2}\sum_{i=1}^{\infty}(\ln x_{i+1})\Delta h(x_i) - \sum_{i=1}^{\infty}\sum_{v=2}^{2k-2}\frac{B_v}{v!}[\Delta \ln^{(v-1)}(x_i)]h^v(x_i) + R'_{2k}$$

where

$$(15)\quad R'_{2k} = \frac{B_{2k}}{2k!}\sum_{i=1}^{\infty}[\ln^{(2k)}(x_i + \theta h(x_i))]h^{2k+1}(x_i)$$

Alternate forms for the remainder can be found. If, for example, we refer to formula (8) we have

$$R'_{2k} = -\theta\frac{B_{2k}}{2k!}\sum_{i=1}^{\infty}(\Delta \ln^{2k-1}x_i)h^{2k}(x_i)$$

Now if we let $h(x_i) = 1$ in (14) and perform the differentiation on $\ln x$ we get the following form.

$$(16)\quad \ln G(x) = (x + \frac{1}{2})\ln x - x + 1 \sum_{v=2}^{2k-1}\frac{B_v}{v(v-1)}\cdot\frac{1}{x^{v-1}} + R_{2k}$$

If we use the second formula for R_{2k} we have

$$(17)\quad R'_{2k} = \theta\,\frac{B_{2k}}{2k(2k-1)}\;\frac{1}{x^{2k-1}}\,.$$

This differs from the classical * formula for $\ln \Gamma(x)$ only by the absence of $\ln \sqrt{2\pi}$ and the presence of 1. However, any general periodic function can be added to a solution of (10) and the result will still be a solution. We consequently add $\ln \sqrt{2\pi} - 1$ to the right member of formula (14). We denote the function that we obtain by $\ln \underset{h}{\Gamma}(x)$. We have

$$\ln \underset{h}{\Gamma}(x) = \ln \sqrt{2\pi} + x \ln x - x + \tfrac{1}{2}(\ln x)h(x)$$

$$+ \frac{1}{2}\sum_{i=1}^{\infty} (\ln x_{i+1}) \Delta h(x_i) + \sum_{i=1}^{\infty} \sum_{v=2}^{2k-2} \frac{B_v}{v!} [(\Delta \ln^{(v-1)} x_i) h^v(x_i)]$$

$+ R'_{2k}$, where R'_{2k} is given by (15) or (17).

We note particularly from (17) that $x^{2k-2} R'_{2k}$ approaches zero when x becomes infinite. Since $\Delta \ln \Gamma(x) = h(x) \ln x$ the

$$\left(\underset{h}{\Gamma}(x + h(x)) = x^{h(x)} \underset{h}{\Gamma}(x) \right)$$

5. Gamma in the Complex Plane

Draw a line parallel to the axis of imaginaries and distant $\varepsilon > 0$ from it. All variables are confined to the half-plane $x \geq \varepsilon$. We assume $0 < \eta < |h(z)| < E$ also $Re(z + h(z)) > 0$. We require that h(z) be real when z = x is real. In addition we assume that

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* See, for example, Fort page 61

$h(z)$ is analytic over the half-plane in question. We also assume that $h(x)$ meets all the requirements previously put upon it. We let $\ln z = \ln\sqrt{x^2+y^2} + \varphi i$ where $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. The points z_i are determined by the equations

$z_1 = z, \quad z_i = z_{i-1} + h(z_{i-1}), \quad i > 1.$

We write down the expression.

$$(1^{\circ}) \quad \ln\sqrt{2\pi} + z\ln z - z + \frac{1}{2}(\ln z)h(z) +$$

$$\frac{1}{2}\sum_{i=1}^{\infty}(\ln z_{i+1})\Delta h(z_i) + \sum_{i=1}^{\infty}\sum_{v=2}^{2k-2}\frac{B_v}{v!}\left[\Delta \ln^{(v-1)}(z_i)\right] h^v(z_i)$$

$$+ R'_{2k}$$

where

$$R'_{2k} = \frac{1}{2k!}\int_0^1 {}_{2k}(t)\left[\sum_{i=1}^{\infty}\ln^{(2k)}(z_i + h(z_i)t)h^{2k+1}(z_i)\right]dt$$

All series appearing here converge uniformly over the half-plane $x \geqq \varepsilon$, the first by assumption and the others by easy proof (See par. 3). Formula (1°) consequently defines an analytic function. This function reduces to $\ln \Gamma_h(x)$ when z is real. This is the same function for different values of k. Suppose that there were two. These two would both be analytic and each would reduce to $\Gamma_h(x)$ when z is real. They are consequently identical from the general theory of analytic functions. We denote the function defined by (18) by $\ln \Gamma_h(z)$. Similarly the relation

$$\frac{\Delta \ln \Gamma_h(z)}{h(z)} = \ln z$$ will hold for

complex z from the general theory of analytic functions.

6. Another Generalization

Let us consider the equation

$$\Delta u(x) = \ln x \tag{19}$$

We write this

$$\frac{\Delta u(x)}{h(x)} = \frac{\ln(x)}{h(x)}$$

Replacing $\frac{\ln x}{h(x)}$ by $F(x)$ we have

$$\frac{\Delta u(x)}{h(x)} = F(x)$$

Now if $h(x)$ is as previously and $F(x)$ is such that $F^{(2k)}$ retains the same sign and $F^{(2k)}(x)\, F^{(2k-2)}(x) > 0$ and $F^{(2k)}(x).x^{2k-2}$ approaches zero when x becomes infinite, then (19) can be solved as was (10). The solution which reduces to $\Gamma(x)$ when $h(x) \equiv 1$ is a generalization of $\Gamma(x)$. The requirements on F place further restrictions on $h(x)$. However, if, for example, $h(x) = \frac{\ln x}{x \ln x - x}$, $x > e^2$ then all requirements are fulfilled. We call this generalization $\overset{h}{\Gamma}(x)$. Clearly

$$\overset{h}{\Gamma}(x + h(x)) = x\,\overset{h}{\Gamma}(x). \tag{20}$$

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